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On the nonrelativistic dynamics of heavy particles near the production threshold

A.A.Pivovarov

Institut für Physik, Johannes-Gutenberg-Universität,

Staudinger Weg 7, D-55099 Mainz, Germany

and

Institute for Nuclear Research of the

Russian Academy of Sciences, Moscow 117312

Abstract

A solution to the Schrödinger equation for the nonrelativistic Green function which is used for describing the heavy quark-antiquark pair production near the threshold in e^+e^- annihilation is presented. A quick comparison with existing results is given. A choice of the effective mass scale for the nonrelativistic system with Coulomb interaction is discussed.

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The hadron production near heavy quark threshold will be thoroughly studied experimentally at future accelerators, e.g. [1]. The dynamics of a slow moving pair of the heavy quark and antiquark near the production threshold is nonrelativistic to high accuracy that justifies the use of the nonrelativistic quantum mechanics as a proper theoretical framework for describing such a system [2]. Being much simpler than the comprehensive relativistic treatment this approach allows one to take into account exactly such essential features of the dynamics as Coulomb interaction [3, 4]. The spectrum of hadronic states produced near the quark-antiquark threshold is contained in the Green function $G(E) = (H - E)^{-1}$ of the effective nonrelativistic Hamiltonian H. In the position space it reads

$$G(E; \mathbf{r}, \mathbf{r}') = \langle \mathbf{r}|(H - E)^{-1}|\mathbf{r}'\rangle \tag{1}$$

and satisfies the Schrödinger equation

$$(H - E)G(E; \mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')\,. \tag{2}$$

The Hamiltonian H is represented in the following general form $H = H_0 + \Delta H$ where the first term H_0 is the Coulomb Hamiltonian

$$H_0 = \frac{p^2}{m} - \frac{C_F \alpha_s}{r} \tag{3}$$

with $p^2 = -\Delta^2$. The color factor for the fermion representation of the gauge group is $C_F = (N_c^2 - 1)/2N_c$, α_s is a QCD coupling constant, m is a mass of the heavy quark. The normalization point for α_s will be fixed later. The second term ΔH accounts for relativistic and perturbative strong interaction contributions which are assumed to be small, i.e. they are treated as corrections to the Coulomb spectrum. The term ΔH has an explicit form

$$\Delta H = \Delta_k V + \Delta_{pot} V + \Delta_{NA} V + \Delta_{BF} V. \tag{4}$$

The quantity $\Delta_k V$ in eq. (4) is a relativistic kinetic energy correction

$$\Delta_k V = -\frac{p^4}{4m^3} \tag{5}$$

the second term $\Delta_{pot}V$ is the strong interaction perturbative corrections to the Coulomb potential [5, 6], the third term $\Delta_{BF}V$ is a Breit-Fermi potential with addition of the color factor C_F [7, 8]. $\Delta_{NA}V = -C_A C_F \alpha_s^2/(2mr^2)$ is the non-Abelian potential of quark-antiquark interaction [9], $C_A = N_c$ is a color factor for the gluons. The corrections to the Coulomb Green function at the origin due to terms $\Delta_k V$, V_{NA} and V_{BF} have first been presented in [10, 11]. Numerous applications of these results and further references can be found in recent literature, e.g. [12, 13, 14, 15, 16]. The treatment of the problem in the present paper is slightly different. For the s-wave production of a quark-antiquark pair in the triplet spin state ($L = 0$, $S = 1$) the Breit-Fermi potential reads [7, 8]

$$\Delta_{BF}V = -\frac{C_F\alpha_s}{2m^2}\left(\frac{1}{r}p^2 + p^2\frac{1}{r}\right) + \frac{11\pi C_F\alpha_s}{3m^2}\delta(\mathbf{r})\,.$$

The correction ΔH can be be rewritten as

$$\Delta H = \Delta_{pot}V - \frac{1}{4m}H_0^2 - \frac{3C_F\alpha_s}{4m}\left[H_0, \frac{1}{r}\right]_+ + \frac{\alpha_s}{m}\left(\frac{5C_F}{4} + \frac{C_A}{2}\right)[H_0, ip_r]_- - \frac{4\pi\alpha_s}{m^2}\left(\frac{C_F}{3} + \frac{C_A}{2}\right)\delta(\mathbf{r}) \tag{6}$$

where $ip_r = \partial_r + 1/r$. Note that a similar form of the representation of the correction in terms of powers of the leading order operator is usual for anharmonic oscillator problems [17]. The part of the potential proportional to the δ-function $\delta(\mathbf{r})$ is a separable potential. The equation for the full GF with such a potential can be exactly solved for the quantity we need. Noticing this fact one rewrites the Hamiltonian in the form

$$H = H_0 + \Delta H = H_{ir} + \alpha_s V \tag{7}$$

with

$$V = -\frac{4\pi}{m^2}\left(\frac{C_F}{3} + \frac{C_A}{2}\right)\delta(\mathbf{r}) = V_0\delta(\mathbf{r})\,. \tag{8}$$

Here H_{ir} is an irreducible Hamiltonian. The new representation for the GF reads

$$G(E) = (H - E)^{-1} = (H_{ir} + \alpha_s V - E)^{-1}\,. \tag{9}$$

Introducing the GF $G_{ir}(E)$ of the irreducible Hamiltonian

$$G_{ir}(E) = (H_{ir} - E)^{-1} \tag{10}$$

one obtains the following equation for the full Green function $G(E)$ (written in the operator form)

$$G(E) = G_{ir}(E) - \alpha_s G_{ir}(E) V G(E)\,. \tag{11}$$

Eq. (11) is exactly solved for the position representation component $G(E;0,0)$ with the result

$$G(E;0,0) = \frac{G_{ir}(E;0,0)}{1 + \alpha_s V_0 G_{ir}(E;0,0)}\,. \tag{12}$$

Eq. (12) accounts for the $\delta(\mathbf{r})$ part of the correction to the irreducible Hamiltonian exactly. It is analogous to the usual representation of the vacuum polarization function through the one particle irreducible block. In this case the irreducible object $G_{ir}(E;0,0)$ is characterized by the absence of $\delta(\mathbf{r})$ interaction in it. The function $G_{ir}(E;0,0)$ can be found perturbatively using the Coulomb solution as a leading order approximation.

The Green function $G(E)$ emerges as a nonrelativistic limit of the relativistic scattering amplitude near the production threshold. The nonrelativistic Hamiltonian can be constructed from the QCD Lagrangian. In the leading order of nonrelativistic expansion there is an energy independent factor (matching coefficient) $C(\alpha_s, m)$ that allows one to map the quantum mechanical quantities onto the relativistic cross section near the threshold. In higher orders of nonrelativistic expansion further terms of the expansion of the current itself and new vertices of the effective Lagrangian are generated that should be accounted for when the cross section in QCD is calculated. We do not discuss these terms here because their contributions start at the higher orders in expansion parameters. Therefore the corresponding expressions should be taken only in the leading order of hard loop expansion. The cross section of the heavy quark-antiquark pair production near the threshold in e^+e^- annihilation contains a part with the nontrivial loop expansion. It has the form

$$R^{th}(s) \sim C(\bar{\alpha}_s, m) \mathrm{Im}\; G(E;0,0)\,, \quad s = (2m + E)^2\,. \tag{13}$$

Here $C(\bar{\alpha}_s, m)$ is the matching (hard or high energy) coefficient, $\bar{\alpha}_s$ is the coupling constant. Generally, one can use the different normalization points (or different subtraction procedures) for $\bar{\alpha}_s$ and for the corresponding coupling constant α_s which enters the expression for the nonrelativistic Green function. The vacuum polarization function near the threshold $C(\bar{\alpha}_s, m)G(E;0,0)$ requires subtraction. This feature is familiar from the PT analysis of the vacuum polarization function in the full theory. To obtain a finite quantity one can differentiate $C(\bar{\alpha}_s, m)G(E;0,0)$ with regard to E (constructing the D-function) or take the discontinuity across the physical cut (constructing the imaginary part or cross section $R^{th}(s)$). Both D-function and $R^{th}(s)$ are finite. We write

$$G(E;0,0) = \frac{1}{\alpha_s V_0} - \frac{1}{\alpha_s V_0} \frac{1}{1 + \alpha_s V_0 G_{ir}(E;0,0)}$$

and obtain the following expression for the D-function

$$C(\bar{\alpha}_s, m) \frac{d}{dE} G(E;0,0) = \frac{C(\bar{\alpha}_s, m)}{(1 + \alpha_s V_0 G_{ir}(E;0,0))^2} \frac{d}{dE} G_{ir}(E;0,0)\,. \tag{14}$$

The imaginary part of $C(\bar{\alpha}_s, m)G(E;0,0)$ reads

$$C(\bar{\alpha}_s, m) \mathrm{Im}\; G(E;0,0) = \frac{C(\bar{\alpha}_s, m) \mathrm{Im}\; G_{ir}(E;0,0)}{(1 + \alpha_s V_0 \mathrm{Re}\; G_{ir}(E;0,0))^2 + (\alpha_s V_0 \mathrm{Im}\; G_{ir}(E;0,0))^2}\,. \tag{15}$$

The quantities in eqs. (14) and (15) are finite. The explicit expression for the coefficient $C(\bar{\alpha}_s, m)$ has been found at the $\bar{\alpha}_s^2$ order within dimensional regularization [18]. It contains a singularity of the form

$$C_S(\bar{\alpha}_s, m) = 1 - C_F \bar{\alpha}_s^2 \left(\frac{C_F}{3} + \frac{C_A}{2} \right) \frac{1}{\varepsilon} \tag{16}$$

where only the singular part $C_S(\bar{\alpha}_s, m)$ of the coefficient $C(\bar{\alpha}_s, m)$ is written. This singularity cancels in eqs. (14) and (15). We consider this cancellation (the renormalization procedure) for the case of D-function only. To fulfill the renormalization procedure we use the dimensionally regularized Green function in the expression for the correction proportional to V_0. It suffices to substitute the pure Coulomb Green function for this purpose. The dimensionally

regularized Coulomb Green function at the origin takes the explicit form [15]

$$G_C^{DR}(\kappa;0,0)=\frac{m}{4\pi}\left\{-\kappa+\frac{C_F\alpha_s m}{2}\left(\frac{1}{\varepsilon}+\ln\frac{\mu^2}{\kappa^2}-2\psi\left(1-\frac{C_F\alpha_s m}{2\kappa}\right)\right)\right\} \tag{17}$$

where $\kappa^2=-mE$, $\psi(z)=\Gamma'(z)/\Gamma(z)$ is digamma function and $\Gamma(z)$ is Euler's Γ-function. One finds

$$(1+\alpha_s V_0 G_{ir}(E;0,0))^2\to(1+\alpha_s V_0 G_C^{DR}(E;0,0))^2=Z^{-1}(1+\alpha_s V_0 G_C(E;0,0))^2 \tag{18}$$

where

$$Z^{-1}=1+C_F\frac{\alpha_s^2}{4\pi}m^2V_0\frac{1}{\varepsilon}=1-C_F\alpha_s^2\left(\frac{C_F}{3}+\frac{C_A}{2}\right)\frac{1}{\varepsilon}\,. \tag{19}$$

The finite (renormalized) Coulomb Green function at the origin has the form

$$G_C(\kappa;0,0)=\frac{m}{4\pi}\left\{-\kappa+\frac{C_F\alpha_s m}{2}\left(\ln\frac{\mu^2}{\kappa^2}-2\psi\left(1-\frac{C_F\alpha_s m}{2\kappa}\right)\right)\right\}\,. \tag{20}$$

The renormalization constant Z cancels the divergence $C_S(\bar\alpha_s,m)$ of the high energy coefficient $C(\bar\alpha_s,m)$ in a proper order of loop expansion. One uses $\bar\alpha_s=\alpha_s+O(\alpha_s^2)$ as a formal PT relation to achieve the cancellation. The finite coefficient of order α_s^2 (or $\bar\alpha_s^2$) depends on particular ways of subtraction in $C(\bar\alpha_s,m)$ and $G_{ir}(E)$. If the subtraction procedures for $C(\bar\alpha_s,m)$ and $G_{ir}(E)$ are not properly coordinated during the calculation the finite coefficient is fixed by matching [19]. If the calculation for both $C(\bar\alpha_s,m)$ and $G_{ir}(E)$ has been done within one and the same subtraction scheme this matching is automatic, e.g. [20]. Therefore after the standard renormalization, eq. (12) gives the representation of $G(E;0,0)$ as a Dyson sum of irreducible terms. The spectrum of the full system is determined by the equation

$$G_{ir}(E;0,0)^{-1}+\alpha_s V_0=0 \tag{21}$$

where $G_{ir}(E;0,0)$ is constructed perturbatively. Eq. (21) can be solved exactly or perturbatively. The isolated roots of eq. (21) give the discrete spectrum of the system. There is also a continuous spectrum given by the discontinuity of $G_{ir}(E;0,0)$ across the cut at positive

values of E. For the continuous spectrum one can use the Coulomb GF in the denominator of eq. (15).

The problem of calculating the near-threshold cross section reduces to the construction of the spectrum of the irreducible Hamiltonian H_{ir}

$$H_{ir} = H_0 + \Delta V_{pot} - \frac{1}{4m}H_0^2 - \frac{3C_F\alpha_s}{4m}\left[H_0, \frac{1}{r}\right]_+ + \frac{\alpha_s}{m}\left(\frac{5C_F}{4} + \frac{C_A}{2}\right)[H_0, ip_r]_- \,. \qquad (22)$$

The spectrum is found within PT. The leading order spectrum is given by the renormalized pure Coulomb solution eq. (20). The term $\Delta_{pot}V$ represents the first and second order perturbative QCD corrections to the Coulomb potential which were studied. The correction due to the first iteration of the $\Delta_{pot}V$ term has been found in ref. [21] where the simple and efficient framework for computing the iterations of higher orders was formulated. The explicit formulas for these corrections can be found in [16]. For the kinetic H_0^2 term one finds

$$\rho_k(E) = \sum_{E'} \delta(E' - \frac{E'^2}{4m} - E)\rho(E') \qquad (23)$$

where $\rho(E)$ is a density of Coulomb states with the energy E

$$\rho(E) = \frac{1}{\pi}\mathrm{Im}\; G_C(E;0,0)\,. \qquad (24)$$

The sum is over the whole spectrum. For the discrete levels it gives

$$\rho_k(E)|_{disc} = \sum_n \delta(E_n - \frac{E_n^2}{4m} - E)|\psi_n(0)|^2 \qquad (25)$$

with ψ_n being a bound state with the energy level E_n. The position of the pole is now at $E_n^{pole} = E_n - E_n^2/4m$. For the continuous spectrum the correction reads

$$\rho_k(E)|_{cont} = \int_0^\infty dE'\delta(E' - \frac{E'^2}{4m} - E)\rho(E') = \frac{1}{1 - \bar{E}/2m}\rho(\bar{E}) \qquad (26)$$

with

$$E - \bar{E} + \frac{\bar{E}^2}{4m} = 0\,. \qquad (27)$$

Eq. (27) can be solved perturbatively for small E. One obtains

$$\bar{E} = E + \frac{E^2}{4m} + O(E^3) \tag{28}$$

which is the substitution used in refs. [10, 11]. Note that there is no correction to wave functions because H_0^2 has no non-diagonal matrix elements between the Coulomb states. The last term in eq. (22) gives no correction to the spectrum. The term with the anticommutator $[H_0, r^{-1}]_+$ in eq. (22) gives the correction to the spectrum which was presented earlier as an energy dependent shift of the coupling constant. This correction can be written in the form

$$G_C(E) + \Delta_a G_C(E) = G_C\left(E; \alpha_s \to \alpha_s\left(1 + \frac{3E}{2m}\right)\right) \tag{29}$$

which is valid at small E and was presented in [10, 11]. Note, however, that while the δ-function part of the correction to the Hamiltonian can be considered unique because it represents a reducible vertex, the irreducible potential H_{ir} can be chosen in different forms. Indeed, one can rewrite the sum of kinetic and anticommutator terms of H_{ir} in the form

$$-\frac{1}{4m}H_0^2 - \frac{3C_F\alpha_s}{4m}\left[H_0, \frac{1}{r}\right]_+ = \frac{5}{4m}H_0^2 - \frac{3}{4m^2}\left[H_0, p^2\right]_+ \tag{30}$$

which is a reshuffling of contributions within the irreducible correction. While the kinetic term H_0^2 has only its coefficient changed as a result of such a reshuffling, the new anticommutator term leads to a modification of the leading order GF of the following form

$$\begin{aligned}
&\frac{1}{H_0 - E} + \frac{3}{4m^2}\left(p^2\frac{1}{H_0 - E} + \frac{1}{H_0 - E}p^2\right) + \frac{1}{H_0 - E}E\frac{3p^2}{2m^2}\frac{1}{H_0 - E} \\
&= \frac{3}{4m^2}\left(p^2\frac{1}{H_0 - E} + \frac{1}{H_0 - E}p^2\right) + \left(\frac{p^2}{m}\left(1 - \frac{3E}{2m}\right) - \frac{C_F\alpha_s}{r} - E\right)^{-1}.
\end{aligned} \tag{31}$$

The first term of this equation does not affect the structure of the spectrum. The last term in eq. (31) can be interpreted as a correction to the mass. To a certain degree this is equivalent to the previous case (the correction to the coupling α_s in eq. (29)) because the genuine parameter of the Coulomb problem is the Bohr radius or momentum, $p_B = \alpha_s m$ (or

$C_F\alpha_s m/2$). One can see that both changes $\alpha_s \to \alpha_s\,(1+3E/2m)$ and $m \to m/\,(1-3E/2m)$ lead to the same result for p_B within the accuracy of the approximation, i.e. up to higher order terms in E

$$p_B \to p_B\left(1+\frac{3E}{2m}\right) = \frac{p_B}{1-\frac{3E}{2m}} + O(E^2)\,. \tag{32}$$

One can check that the total correction to the discrete energy levels, for instance, is the same. The old solution eq. (29) gives

$$m\Delta E_n = -\frac{1}{4}E_n^2 + 3E_n^2 = \frac{11}{4}E_n^2 \tag{33}$$

while the new decomposition eq. (31) results in

$$m\Delta E_n = \frac{5}{4}E_n^2 + \frac{3}{2}E_n^2 = \frac{11}{4}E_n^2\,. \tag{34}$$

This coincidence is valid only parametrically in the region of applicability of perturbation theory in the expansion parameters E/m and/or α_s. It can result in numerical difference when extrapolated to larger energies in the continuous spectrum. The different forms of the decomposition may lead to different numerical predictions within numerical evaluation [10, 11, 12, 22, 23].

The expression given in eq. (12) has a standard structure of the polarization function. The solution E_f to eq. (21) (one or the first of the poles of the generalized "propagator" of the full system with the Hamiltonian H) is an important dimensional parameter for the system. For the observables which are saturated with the contributions of the discrete spectrum the quantity E_f can serve as a natural mass parameter. In this case the position of a pole (the numerical value of some solution E_f to eq. (21)) can be chosen as a scale for the system instead of the heavy quark pole mass. However, for the observables which are saturated with the contributions of the continuous spectrum or have a considerable admixture of such contributions, the natural mass scale is not necessarily related to E_f and is determined by other properties of the full interaction.

To conclude, we have presented a solution to the Schrödinger equation for the nonrelativistic Green function. The solution has the form of a resumed geometric series (Dyson resummation) of irreducible blocks which is usual for quantum field theory. The property of irreducibility is defined with respect to the δ-function part of the interaction potential. Different decompositions of the irreducible Hamiltonian H_{ir} for the treatment within PT are considered. A new form of the first order correction to the spectrum of irreducible Hamiltonian H_{ir} is given.

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